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DETERMINING THE SHEAR MODULUS OF WATER IN EXPERIMENTS WITH A FLOATING DISK

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The problem of decaying rotation of a disk floating on the surface of a viscoelastic fluid in a cylindrical container is solved by numerical methods. The motion is found to have the form of decaying oscillations observed previously for water. In addition to the viscosity coefficient, the constructed mathematical model of the viscoelastic fluid has two more independent parameters: shear modulus and time of relaxation of elastic stresses. Elastic parameters of water are determined through comparisons with experimental data.

Key words: *non-Newtonian fluids, rheology, viscoelastic fluids, viscosimetric measurements.*

Introduction. Rotational motion of a floater made in the form of a disk floating on the fluid surface in a cylindrical container is studied in the present work by numerical methods (Fig. 1). The disk is instantaneously set into motion (rotation around its own centerline coinciding with the container centerline). When the disk acquires steady-state rotation, the action of the external moment is terminated, and the disk motion starts to decay owing to viscous friction on the fluid. It was found in the experiments [1] that the decay has an oscillating character (Fig. 2). Based on this fact, water may be assumed to have viscoelastic properties, though it is normally considered as a Newtonian fluid. The problem considered here was numerically solved in [2] in the approximation of the Newtonian fluid; changes in the initial conditions and physical parameters of the disk and the container were found to initiate no backward motion of the disk. The data of [1] are interpreted in the present work on the basis of several linear models of a viscoelastic fluid. The most suitable model turned out to be the model of viscoelasticity [3] with an exponential function of decaying of elastic stresses. As compared with the Newtonian fluid model, the model developed in [3] has two more independent parameters: shear modulus G and time of relaxation of elastic stresses T. The fluid considered is interpreted as a substance with the stress dependent on the strain and strain rate. In turn, the strain is determined by the differences in the material configuration at consecutive times.

Mathematical Model. The stress tensor of the examined fluid is presented as $\sigma = \sigma_{\rm vis} + \sigma_{\rm el}$. The elastic part of the stress tensor of the isotropic fluid with the convective elasticity

$$\sigma_{\rm el}(t'') = \int\limits_{-\infty}^{t''} \exp\Big(-\frac{t''-t'}{T}\Big) G\, d\varepsilon(t')$$

is used in the discrete form

$$\sigma_{\rm el}(t'') = G\varepsilon + \sigma_{\rm el}(t') \exp\left(-(t''-t')/T\right)$$

(ε is the tensor of small strains in the time interval $t''-t'$). It is taken into account that elastic stresses are not additive in each time interval and that relaxation occurs [4]. The viscous part of the stress tensor satisfies the Newton equation $\sigma_{\rm vis} = -\eta\dot\varepsilon$, where $\dot\varepsilon$ is the strain-rate tensor.

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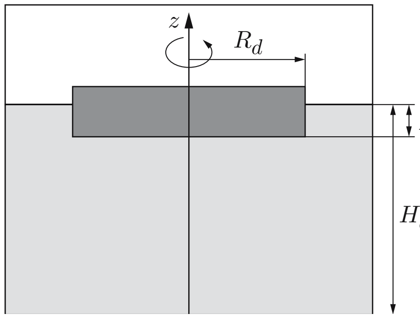

Fig. 1

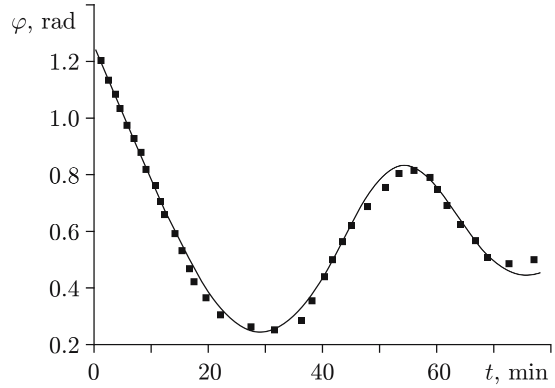

Fig. 2

Fig. 1. Arrangement of the experiment.

Fig. 2. Angle of disk rotation versus the time after termination of the external rotating moment, for decaying motion: the curve and the points are the calculated and experimental results [1], respectively.

In dimensionless variables, with the linear sizes normalized to the inner radius R_c of the cylindrical container, velocity to ν/R_c, pressure perturbation with respect to the quantity $\rho g z$ to $\rho\nu^2/R_c^2$ (ν is the kinematic viscosity and ρ is the fluid density), time to R_c^2/ν, and angular velocity of the disk to ν/R_c^2; the motion of the disk and the fluid surrounding the disk is described by the following system of equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} = \Delta \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla p - \boldsymbol{f}_{\rm el}; \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0; \tag{2}$$

$$\frac{d\omega_d}{dt} = M_{\rm vis} + M_{\rm el}^t. \tag{3}$$

Here t is the time, $\boldsymbol{u} = (u_r, u_\theta, u_z)$ is the velocity in the cylindrical coordinate system (r, θ, z), p is the pressure perturbed with respect to the hydrostatic pressure, ω_d is the angular velocity of disk motion, and $\boldsymbol{f}_{\rm el}$ is the density of the elastic force in the medium. Equation (1) is the law that describes the behavior of momentum for the fluid particles; Eq. (2) is the condition of fluid incompressibility; Eq. (3) describes the decaying motion of the disk. The dimensionless bulk density of the elastic force at each time step is explicitly determined by the formula

$$\boldsymbol{f}_{\rm el}^{t+\delta t} = E\Delta \boldsymbol{u} + \boldsymbol{f}_{\rm el}^t \exp{(-\delta t/\tau)} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{f}_{\rm el}^t \exp{(-\delta t/\tau)}\delta t, \tag{4}$$

where $E = GR_c^2\delta t/(\rho\nu^2)$. The first term in Eq. (4) is the increment of elastic stresses resulting from displacements that occurred during the time δt equal to the time step of the difference scheme. The second term describes the exponential relaxation of elastic stresses in a fluid element by the time t. The third term reflects the convective transport of the density of the elastic force during the time δt.

The expressions for the moments acting on the disk from the viscoelastic fluid contain two terms: the moment acting on the lower surface of the disk and the moment acting on the submerged part of its side surface. The expressions for these moments in the cylindrical coordinates have the form

$$M_{\rm vis} = \frac{2\pi\rho R_c^5}{I_d}\Big(- \int\limits_0^{R_d/R_c} \Big(\frac{\partial u_\theta}{\partial z}\Big)\Big|_{z=(H_c-H_d)/R_c} r^2\,dr + \int\limits_{(H_c-H_d)/R_c}^{H_c/R_c} \Big(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\Big)\Big|_{r=R_d} dz\Big),$$

$$M_{\mathrm{el}}^{t+\delta t} = \frac{2\pi R_c^7 G\,\delta t}{I_d \nu^2}\Big(-\int\limits_0^{R_d/R_c} \Big(\frac{\partial u_\theta}{\partial z}\Big)\Big|_{z=(H_c-H_d)/R_c} r^2\,dr$$

$$+ \int\limits_{(H_c-H_d)/R_c}^{H_c/R_c} \Big(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\Big)\Big|_{r=R_d} dz\Big) + M_{\mathrm{el}}^t \exp\Big(-\frac{\delta t}{\tau}\Big),$$

where R_d is the disk radius, I_d is the moment of inertia of the disk, and H_c and H_d is the height of the fluid in the cylindrical container and the height of the disk, respectively.

The boundary conditions on the solid walls are the no-slip conditions for the fluid:

$$\begin{aligned}
z = 0,\quad 0 < r < R_c&: \qquad u_z = 0,\quad u_r = 0,\quad u_\theta = 0,\\
z = H_c,\quad 0 < r < R_d&: \qquad u_z = 0,\quad u_r = 0,\quad u_\theta = \omega_d r,\\
0 < z < H_c,\quad r = R_c&: \qquad u_z = 0,\quad u_r = 0,\quad u_\theta = 0,\\
H_c - H_d < z < H_c,\quad r = R_d&: \qquad u_z = 0,\quad u_r = 0,\quad u_\theta = \omega_d R_d.
\end{aligned} \tag{5}$$

Under the assumption that the free boundary of the fluid is horizontal, there are no tangential stresses on this boundary:

$$z = H_c,\quad R_d < r < R_c: \qquad u_z = 0,\quad \frac{\partial u_r}{\partial z} = 0,\quad \frac{\partial u_\theta}{\partial z} = 0. \tag{6}$$

The following conditions are satisfied in the axisymmetric approximation at the center of the container:

$$u_r = u_\theta = 0, \qquad \frac{\partial u_z}{\partial r} = \frac{\partial p}{\partial r} = 0. \tag{7}$$

At the initial time, the fluid is at rest, and the disk is instantaneously set into steady-state motion with an angular velocity ω_d. At the first stage, Eqs. (1) and (2) with the boundary conditions (5)–(7) are calculated. When the steady-state regime of disk motion is reached, the action of the external moment is terminated. At this stage, Eqs. (1)–(7) are calculated on the basis of the steady-state distribution of velocities and pressures obtained at the first stage. The angle of turning of the disk is determined by the formula $\varphi(t + \delta t) = \varphi(t) + \omega_d\,\delta t$.

Method of the Solution. The numerical solution was found by the finite-difference method. Uniform spatial grids with the size 32×32 in the axial and radial directions were used. Discretization of derivatives in hydrodynamic equations was performed by the central-difference scheme with accuracy to $(\delta z)^2$ for spatial variables and by the one-sided difference scheme with accuracy to δt for the time variable. The resultant nonlinear algebraic equations were linearized by the Newton scheme, and the linearized systems of equations were solved at each time step by the Gaussian method of elimination [5].

Results and Discussion. The calculations were performed for a cylindrical container with the following sizes [1]: container radius $R_c = 40$ cm, its height $H_c = 10$ cm, radius of the cylindrical floater (disk) $R_d = 20$ cm, and moment of inertia $I_d = 4.5\cdot 10^{-3}$ kg $\cdot$ m^2. The disk was placed in the center of the container and was submerged to a depth $H_d = 2$ cm. The fluid density and viscosity were equal to those of water. The initial angular velocity of disk rotation $\omega_d^0 = 7.81\cdot 10^{-4}$ sec^{-1} was determined by means of numerical differentiation of the experimental dependence of the angle of turning of the disk on time (see Fig. 2).

The challenge of the present activities was to choose the elastic parameters of water G and T for the best fitting between the experimental and calculated dependences of the angle of turning of the disk on time. Figure 2 shows the curve calculated for $G = 1.3\cdot 10^{-5}$ Pa and $T = 29.33$ min. It is seen that the numerical results are in good agreement with the data of [1], at least for the first wave of oscillations.

There are some experimentally obtained values of the shear modulus for various media: $G = 10^4$ Pa for gelatin jelly [6], $G = 2$–3 Pa for a 5-% suspension of starch in mineral oil [7], $G = 10^{-1}$–10^{-3} Pa for blood [8], and $G = 1.3\cdot 10^{-5}$ for water (data of the present work). Thus, the shear modulus of water is at least two orders lower than the shear modulus of blood, which is the least "elastic" one among viscoelastic fluids.

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